

ASYMPTOTIC ANALYSIS OF ORTHOGONAL POLYNOMIALS VIA THE TRANSFER MATRIX APPROACH

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ABSTRACT. In this paper, we present a new method via the transfer matrix approach to obtain asymptotic formulae of orthogonal polynomials with asymptotically identical coefficients of bounded variation. We make use of the hyperbolicity of the recurrence matrices and employ Kooman's Theorem to diagonalize them simultaneously. The method introduced in this paper allows one to consider products of matrices such that entries of consecutive matrices are of bounded variation.

Finally, we apply the asymptotic formulae obtained to solve the point mass problem on the real line when the measure is essentially supported on an interval. We prove that if a point mass is added to such a measure outside its essential support, then the perturbed recurrence coefficients will also be asymptotically identical with the same limit and of bounded variation.

1. INTRODUCTION

Let μ be a non-trivial measure on \mathbb{R} such that for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} |x|^n d\mu(x) < \infty. \quad (1.1)$$

We form an inner product and a norm on $L^2(\mathbb{R}, d\mu)$ as follows: for any $f, g \in L^2(\mathbb{R}, d\mu)$, define

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) d\mu(x), \quad (1.2)$$

$$\|f\|^2 = \int_{\mathbb{R}} f(x)^2 d\mu(x). \quad (1.3)$$

By the Gram-Schmidt process, we can orthogonalize $1, x, x^2, \dots$ and obtain the family of orthogonal polynomials. We denote the n -th monic orthogonal polynomial as $P_n(x)$ and the n -th orthonormal polynomial as $p_n(x)$. Let

$$\kappa_n = \frac{1}{\|P_n\|}. \quad (1.4)$$

Then the n -th orthonormal polynomial is given by

$$p_n(x) = \frac{P_n(x)}{\|P_n\|} = \kappa_n x^n + \text{lower order terms}. \quad (1.5)$$

It is well-known that these orthogonal polynomials satisfy the following three-term recurrence relations:

$$xP_n(x) = P_{n+1}(x) + b_{n+1}P_n(x) + a_n^2P_{n-1}(x) \quad (1.6)$$

$$xp_n(x) = a_{n+1}(x)p_{n+1}(x) + b_{n+1}p_n(x) + a_np_{n-1}(x) \quad (1.7)$$

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with the properties that

$$a_n = \frac{\|P_n\|}{\|P_{n-1}\|} = \frac{\kappa_{n-1}}{\kappa_n} > 0 \quad \text{and} \quad b_{n+1} = \langle xp_n, p_n \rangle. \quad (1.8)$$

The $(a_n, b_n)_{n=1}^\infty$ are called the recurrence coefficients associated to the measure $d\mu$. The recurrence relation (1.7) is often represented by the matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (1.9)$$

which is called the **Jacobi matrix**.

Apart from the Jacobi matrix, there is another representation of the recurrence relation by means of the **transfer matrix** $T_n(x)$: observe that the recurrence relation (1.7) can be rewritten as

$$a_{n+1}p_{n+1}(x) = (x - b_{n+1})p_n(x) - a_n p_{n-1}(x) \quad \text{for } n \geq 0. \quad (1.10)$$

Therefore, the general solution of (1.10) above can be expressed in the following way:

$$\begin{pmatrix} p_{n+1}(x) \\ a_n p_n(x) \end{pmatrix} = A_{n+1}(x) \begin{pmatrix} p_n(x) \\ a_{n-1} p_{n-1}(x) \end{pmatrix}, \quad (1.11)$$

where

$$A_j(x) = a_j^{-1} \begin{pmatrix} x - b_j & -1 \\ a_j^2 & 0 \end{pmatrix}. \quad (1.12)$$

This motivates the definition of the **transfer matrix**

$$T_n(x) = A_n(x)A_{n-1}(x) \cdots A_1(x) \quad \text{for } n \geq 1. \quad (1.13)$$

In particular, the recurrence relation could be expressed in terms of the transfer matrix applied to $(1, 0)^T$:

$$\begin{pmatrix} p_n(x) \\ a_{n-1} p_{n-1}(x) \end{pmatrix} = T_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (1.14)$$

Moreover, since $\det A_j(x) = 1$ for all $j \geq 1$,

$$\det T_n(x) = \prod_{j=1}^n \det A_j(x) = 1. \quad (1.15)$$

The transfer matrix $T_n(x)$ and the $A_n(x)$'s will serve as important tools when we derive the asymptotic formulae for the $p_n(x)$'s in Section 4.

The reader should be reminded that besides the definitions given for $T_n(x)$ and $A_n(x)$ in (1.13) and (1.12), there are a few other commonly used definitions in the literature.

Let J_n be the truncated $n \times n$ matrix obtained from the first n rows and columns of the Jacobi matrix J . Note that by the recurrence relation,

$$(J_n - x) \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-2}(x) \\ p_{n-1}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -a_n p_n(x) \end{pmatrix}. \quad (1.16)$$

Every zero of $p_n(x)$ is an eigenvalue of J_n and the orthogonal polynomials form the eigenfunction for the truncated Jacobi matrix. In other words, $p_n(x)$ is the characteristic polynomial of J_n . Moreover, it is known that the spectral theory of one-dimensional operators (eg., the one-dimensional discrete Schrödinger operator) has a lot in common with the classical theory of orthogonal polynomials on the real line. Many results were proven for the Jacobi operator and the Schrödinger

operator in parallel (see for example [8]). Therefore, orthogonal polynomials have gained the attention of both the spectral theory community and the Schrödinger operator community in recent years.

For a more comprehensive introduction to the theory of orthogonal polynomials on the real line, the reader may refer to [4, 8, 17, 23].

2. RESULTS

First, we consider a measure in the **Nevai class** $\mathcal{M}(a, b)$, which consists of measures with recurrence coefficients satisfying $a_n \rightarrow a, b_n \rightarrow b$. It is well-known that measures in $\mathcal{M}(a, b)$ have essential support $[b - 2a, b + 2a]$ (see [17] for a detailed discussion). In this paper, we limit ourselves to $a_n \rightarrow a \neq 0$ (Dombrowski [6] showed that a Jacobi matrix with $\liminf |a_n| = 0$ has empty a.c. spectrum). For the importance of the Nevai class and references to the many investigations thereof, the reader may refer to [15, 16, 17, 18].

Apart from the asymptotic formulae listed in Theorem 2.1 below, it is interesting to note the method developed in this paper by means of applying Kooman's Theorem to the transfer matrix (see Section 4). We make use of the fact that if x_0 is outside the essential support of the measure, then $A_n(x_0)$ is hyperbolic for all large n . This fact allows us to apply Kooman's theorem to simultaneously diagonalize A_n and deduce asymptotic formulae for $p_n(x)$ outside the essential support. Due to the length of the proof, a sketch of the proof is provided in Section 4.1.

The first result can be summarized as follows:

Theorem 2.1. *Let μ be a measure in $\mathcal{M}(a, b)$ with recurrence coefficients of bounded variation and $x_0 \in \mathbb{R} \setminus [b - 2a, b + 2a]$. Then the asymptotic formulae for $p_n(x_0)$ are as follows:*

- (1) $\mu(x_0) > 0$ if and only if given any $\epsilon > 0$, there exists a constant C_ϵ such that

$$|p_n(x_0)| \leq C_\epsilon (\lambda^- + \epsilon)^n. \quad (2.1)$$

where $|\lambda^-| < 1$ is the eigenvalue of $A_\infty(x_0)$ (see (4.1) for the definition of A_∞).

- (2) $\mu(x_0) = 0$ if and only if for every $n \in \mathbb{N}$, $p_n(x_0)$ is in the form

$$p_n(x_0) = \left(\prod_{j=1}^n \lambda_j^+ \right) k_n, \quad (2.2)$$

where $(k_n)_n$ is a convergent sequence of bounded variation that varies according to initial conditions, and λ_j^+ is the eigenvalue of the recurrence matrix $A_j(x)$ whose norm is great than 1 (see (1.12) for the definition of the recurrence matrix $A_j(x)$). The sequence will be computed explicitly in the proof in Section 4.

Then we apply the asymptotic formulae obtained to solve the point mass problem. We add a pure point $x_0 \in \mathbb{R}$ to μ to form the measure $\tilde{\mu}$ as follows:

$$\tilde{\mu} = \mu + \gamma \delta_{x_0}, \quad \gamma > 0. \quad (2.3)$$

In Theorem 2.2 below, we give formulae relating the orthogonal polynomials and the recurrence coefficients of μ and $\tilde{\mu}$. Even though these formulae are known (see [17]), the proofs are given below for the convenience of the reader.

A note on notation. We shall denote objects associated to the measure $\tilde{\mu}$ with a \sim on top. For example, the n -th monic orthogonal polynomial with respect to the measure $\tilde{\mu}$ is denoted as $\tilde{P}_n(x)$.

Theorem 2.2. *Let μ and $\tilde{\mu}$ be measures defined as in (2.3). Then the n -th monic orthogonal polynomials of μ and $\tilde{\mu}$ are related by the following formula:*

$$\tilde{P}_n(x) = \left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 \left[P_n(x) - \frac{\gamma P_n(x_0) K_n(x, x_0)}{1 + \gamma K_n(x_0, x_0)} \right], \quad (2.4)$$

with

$$\left(\frac{\kappa_n}{\tilde{\kappa}_n}\right)^2 = \frac{1 + \gamma K_n(x_0, x_0)}{1 + \gamma K_{n-1}(x_0, x_0)}, \quad (2.5)$$

and the reproducing kernel $K_n(x, y)$ is defined as

$$K_n(x, y) = \sum_{j=0}^n p_j(x) p_j(y). \quad (2.6)$$

Furthermore, the recurrence coefficients of μ and $\tilde{\mu}$ are related as follows:

(1)

$$\tilde{a}_n = a_n \sqrt{\frac{t_{n-1}}{t_n}} > 0 \quad (2.7)$$

where

$$t_n = \frac{1 + \gamma K_{n-1}(x_0, x_0)}{1 + \gamma K_n(x_0, x_0)}. \quad (2.8)$$

(2)

$$\tilde{b}_{n+1} = b_{n+1} - \frac{\gamma P_n(x_0) p_{n-1}(x_0) \kappa_{n-1}}{1 + \gamma K_{n-1}(x_0, x_0)} + \frac{\gamma P_{n+1}(x_0) p_n(x_0) \kappa_n}{1 + \gamma K_n(x_0, x_0)}. \quad (2.9)$$

In Section 7 we combine those asymptotic formulae with Theorem 2.2 to prove the following result:

Theorem 2.3. *Let μ be a non-trivial measure on \mathbb{R} such that its recurrence coefficients satisfy*

$$a_n \rightarrow a \neq 0, \quad b_n \rightarrow b; \quad (2.10)$$

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n| + |b_{n+1} - b_n| < \infty. \quad (2.11)$$

The essential support of μ is $[b - 2a, b + 2a]$. If we add finitely many distinct pure points $x_1, \dots, x_k \in \mathbb{R} \setminus [b - 2a, b + 2a]$ to μ as follows

$$\mu_k = \mu + \sum_{j=1}^k \gamma_j \delta_{x_j}, \quad \gamma_j > 0, \quad (2.12)$$

then the recurrence coefficients of μ_k satisfy (2.10) and (2.11).

3. ASYMPTOTIC ANALYSIS AND THE POINT MASS PROBLEM

The point mass problem has a very long history (see the Introduction of [26] for details) and it has its physical significance. As noted in the Introduction, results were often proven for the Jacobi operator and the Schrödinger operator in parallel. As a result, the point mass problem has been investigated by both the orthogonal polynomials and the mathematical physics communities.

The earliest work related to the point mass problem could be due to Wigner-von Neumann [24], where they constructed a potential with an embedded eigenvalue. In 1946, Borg [1] proved a well-known result concerning the Sturm–Liouville problem, which implies that if two Sturm–Liouville operators have spectra differing by a finite number of eigenvalues, then their corresponding potential functions might not be the same. Later, Gel’fand–Levitan [7] showed that in order to recover the potential one also needs the norming constants, which correspond to the weights of pure points in the context of orthogonal polynomials.

The point mass problem has been considered under various settings. In [25] Szwarz considered a measure with bounded support $S \subset [0, +\infty)$ of which the recurrence coefficients satisfy $a_n a_{n+1}^{-1} \rightarrow 1$ and $a_n/b_n \rightarrow \sqrt{A}$ for some $A \geq 0$. It was proven that if a point mass is added to the measure, then

$a_n - a'_n \rightarrow 0$ and $b_n - b'_n \rightarrow 0$ as $n \rightarrow \infty$, where a'_n, b'_n are the perturbed coefficients, though the specific rate of convergence was not shown.

Szwarc's paper also discusses the growth of orthogonal polynomials for measures supported on $[0, 1]$ with a finite number of pure points, with $a_n \rightarrow 1$ and $b_n \rightarrow 1/2$. It was proven under such conditions,

$$\limsup_n |p_n(x)|^{1/n} \leq 1 \quad (3.1)$$

uniformly on $[0, 1]$. Unfortunately, such results (mostly derived from [19]), like many existing asymptotic results on $p_n(x)$, are insufficient to prove Theorem 2.2.

In [16], Máté–Nevai–Totik considered a measure in the Nevai class with coefficients satisfying $a_n \rightarrow 1, b_n \rightarrow 0$ and

$$\sum_n |a_{n+1} - a_n| + |b_{n+1} - b_n| < \infty. \quad (3.2)$$

This measure is supported on $[-1, 1]$. This is a special case among those considered in Theorem 2.3. The authors gave asymptotic formulae for $p_n(x)$ on a compact set $K \subset \mathbb{C}$ with $K \cap [-1, 1] = \emptyset$ as well as on a compact set $K' \subset \text{supp}(\mu) \setminus [-1, 1]$. The authors also presented ratio asymptotic results concerning $p_n(x)/p_{n-1}(x)$. However, in both cases, we need more precise bounds on error terms to establish Theorem 2.3 (see Section 4).

There is an analog of the point mass problem on the unit circle, but the problem is different in nature as the recurrence coefficients of a non-trivial probability measure on $\partial\mathbb{D}$ form a one-parameter family $(\alpha_n)_{n=0}^\infty$ with $\alpha_n \in \mathbb{D}$ (also known as the Verblunsky coefficients in [20, 21]).

The problem on the unit circle reads as follows: let ν be a non-trivial probability measure on $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. One adds a pure point $\zeta \in \partial\mathbb{D}$ to ν to form the new probability measure

$$\tilde{\nu} = (1 - \gamma)\nu + \gamma\delta_\zeta, \quad 1 > \gamma > 0. \quad (3.3)$$

Then one has the following classic result relating the original and the perturbed orthogonal polynomials, which is an analog of (2.4) in Theorem 2.2.

Theorem 3.1. (*Geronimus* [11, 12]) *Suppose the probability measure $\tilde{\nu}$ is defined as in (3.3). Then the n -th monic orthogonal polynomial of $\tilde{\nu}$ is given by*

$$\Phi_n(z, \tilde{\nu}) = \Phi_n(z) - \frac{\varphi_n(z)K_{n-1}(z, \zeta)}{(1 - \gamma)\gamma^{-1} + K_{n-1}(\zeta, \zeta)} \quad (3.4)$$

where $\Phi_n(z)$ and $\varphi_n(z)$ are the monic and normalized orthogonal polynomials of ν respectively; and $K_n(z, \zeta) = \sum_{j=0}^n \varphi_j(\zeta)\varphi_j(z)$.

The point mass problem on the unit circle was further investigated by Cachafeiro–Marcellán [2] and Simon (see Chapter 10.13 of [21]) using very different approaches. In [27] Wong applied the Christoffel–Darboux formula to Simon's result and proved the point mass formula, which shows that the recurrence coefficients $(\alpha_n)_{n=0}^\infty$ of $d\nu$ and $d\tilde{\nu}$ are related as $\alpha_n(d\tilde{\nu}) = \alpha_n(d\nu) + \Delta_n(\zeta)$, where

$$\Delta_n(\zeta) = \frac{(1 - |\alpha_n|^2)^{1/2} \overline{\varphi_{n+1}(\zeta)} \varphi_n^*(\zeta)}{(1 - \gamma)\gamma^{-1} + K_n(\zeta, \zeta)}; \quad K_n(\zeta, \zeta) = \sum_{j=0}^n |\varphi_j(\zeta)|^2. \quad (3.5)$$

The reader may compare (3.5) with the formulae for \tilde{a}_n and \tilde{b}_n in Theorem 2.2.

In [26, 28, 29] Wong studied the point mass problem on the unit circle. In particular, the class W_p consisting of measures of which the Verblunsky coefficients are of p -generalized bounded variation was identified. It was proven that upon adding a point mass to a measure in W_p , we obtain a new probability measure in W_{p+1} . Inductively, we can add a finite number k of distinct pure points to a measure in W_p one after another and we will end up with a new probability measure in W_{p+k} . These results on the unit circle could be read in parallel to Theorem 2.3.

4. THE TRANSFER MATRIX APPROACH

4.1. Structure of this section. In Section 4.2 we consider the recurrence relation in matrix form. Since x_0 is not in the support of the measure, $A_j(x_0)$ is hyperbolic for all large j . Because of the hyperbolicity and the fact that $A_j \rightarrow A_\infty$, we can apply Kooman's Theorem to prove that there is an analytic function U on a neighborhood of A_∞ such that for all large j , A_j can be simultaneously diagonalized (see Section 4.3).

In Section 4.4, we consider a representation that stems out from the diagonalization. In Proposition 4.2), we show a trichotomy based on that representation and prove Theorem 2.1 for two of the three cases, which are special cases that are easier to handle.

The remaining case, being the most difficult one, will be treated in Section 4.5. We prove several estimates for this particular case. In Section 4.6, we combine those estimates to obtain asymptotic formulae for $p_n(x_0)$.

A diagram summarizing the results is provided at the end of the section for the convenience of the reader.

From now on, let μ be a measure in $\mathcal{M}(b, a)$ with recurrence coefficients satisfying (2.10) and (2.11).

4.2. The recurrence relation and $p_n(x_0)$. Note that under the conditions that $a_n \rightarrow a \neq 0$ and $b_n \rightarrow b$,

$$\lim_{j \rightarrow \infty} A_j(x) = A_\infty(x) := a^{-1} \begin{pmatrix} x - b & -1 \\ a^2 & 0 \end{pmatrix}. \quad (4.1)$$

$A_\infty(x)$ has eigenvalues

$$\lambda^\pm = \frac{(x - b) \pm \sqrt{(x - b)^2 - 4a^2}}{2a}. \quad (4.2)$$

Therefore, $A_\infty(x)$ has distinct eigenvalues in \mathbb{R} if and only if $x \in \mathbb{R} \setminus [b - 2a, b + 2a]$. In that case, one of which has absolute value strictly greater than 1 and the other strictly less than 1. We say that $A_\infty(x)$ is **hyperbolic**.

Now consider a fixed point $x_0 \in \mathbb{R} \setminus [b - 2a, b + 2a]$. Without loss of generality, we assume $x_0 > b + 2a$. By (4.2), we have

$$\lambda_n^+ > 1 > \lambda_n^- \quad (4.3)$$

for all large n .

From now on we will write $A_j(x_0)$ as A_j and similarly for other objects that appear in the proof.

4.3. Hyperbolicity of A_j and Kooman's Theorem. Since $A_j \rightarrow A_\infty$, A_j is also hyperbolic for all large j . This allows us to use the following result by Kooman. Adapted to suit the context of this paper, the theorem reads as follows:

Theorem 4.1 (Kooman [13, 14]). *Let A be an $\ell \times \ell$ matrix with distinct eigenvalues. Then there exist $\epsilon > 0$ and analytic functions $U(B)$ and $D(B)$ defined on $S_\epsilon = \{B : \|B - A\| < \epsilon\}$ such that*

(1) $B = U_B D_B U_B^{-1}$, D_B commutes with A .

(2) U_B is invertible for all $B \in S_\epsilon$.

(3) $U_A = 1$, $D_A = A$.

(4) By picking a basis such that A is diagonal, we can have all D_B diagonal with entries being the eigenvalues of B .

Remarks:

- (1) The formulation of Theorem 4.1 is similar to Theorem 12.1.7 of [21], except that in [21] the statement was meant for quasi-unitary matrices. In fact, the proof holds as long as A has distinct eigenvalues.

- (2) *Kooman's Theorem first appeared in Theorem 1.3 of [13]. The first application of Kooman's Theorem to orthogonal polynomials was made by Golinskii–Nevai [10] for the unit circle case. They proved that if the recurrence coefficients of the measure μ on $\partial\mathbb{D}$ satisfy $\alpha_n \rightarrow 0$ and if $\sum_n \|A_{n+1} - A_n\| < \infty$, then the a.c. part of the measure is positive almost everywhere on $\partial\mathbb{D}$.*

Let G be the matrix that diagonalizes A_∞ , i.e.,

$$A_\infty = G^{-1}DG, \quad (4.4)$$

where

$$D = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} \quad (4.5)$$

and λ^\pm are the eigenvalues of A_∞ (defined in (4.2)). Pick a basis in which A_∞ is diagonal. Then by the construction of the function D in Theorem 4.1 above, there exists an integer

$$N > N(\epsilon) \quad (4.6)$$

such that A_j is in some S_ϵ neighborhood of A_∞ and D_{A_j} is a diagonal hyperbolic matrix under this basis for all $j \geq N$. In other words, there exist diagonal matrices

$$D_j = \begin{pmatrix} \lambda_j^+ & 0 \\ 0 & \lambda_j^- \end{pmatrix} \quad (4.7)$$

such that $D_{A_j(x_0)} = GD_jG^{-1}$ and the eigenvalues λ_j^\pm satisfy

$$\lambda_j^+ > 1 > \lambda_j^-, \quad \lim_{j \rightarrow \infty} \lambda_j^\pm = \lambda^\pm. \quad (4.8)$$

In fact, by (1.12) and a straightforward computation, we can show that λ_j^\pm are roots of the characteristic polynomial

$$z^2 - (x_0 - b_j)z + a_j^2 = 0. \quad (4.9)$$

Next, we define

$$G_j = U_{A_j}G. \quad (4.10)$$

Then for $j \geq N$, the matrix A_j can be expressed as

$$A_j = G_j D_j G_j^{-1}. \quad (4.11)$$

4.4. Representation of the recurrence relation. Let E be an integer, we will choose E later in the proof (see Proposition 4.2).

For $n \geq E \geq N$, the transfer matrix applied to the basis vector gives

$$T_n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = G_n D_n G_n^{-1} G_{n-1} D_{n-1} G_{n-1}^{-1} \cdots D_{E+1} G_{E+1}^{-1} G_E v_E, \quad (4.12)$$

where

$$v_E = \begin{pmatrix} v_1^{(E)} \\ v_2^{(E)} \end{pmatrix} := D_E G_E^{-1} A_{E-1} \cdots A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (4.13)$$

Following (4.12), we consider the equation

$$D_n G_n^{-1} G_{n-1} D_{n-1} G_{n-1}^{-1} \cdots D_{E+1} G_{E+1}^{-1} G_E v = L_n \begin{pmatrix} u_n v_1^{(E)} \\ w_n v_2^{(E)} \end{pmatrix}, \quad (4.14)$$

where u_n and w_n are defined implicitly by (4.14) above and

$$L_n = \prod_{k=E+1}^n \lambda_k^+ \quad \text{for } n > E. \quad (4.15)$$

Also, we define

$$\begin{pmatrix} v_n^{(1)} \\ v_n^{(2)} \end{pmatrix} = v(n) = \text{L.H.S. of (4.14)} = L_n \begin{pmatrix} u_n v_1^{(E)} \\ w_n v_2^{(E)} \end{pmatrix}. \quad (4.16)$$

Proposition 4.1. *For any $E \geq N$, either $v_1^{(E)}$ or $v_2^{(E)}$ in (4.13) is non-zero.*

Proof. Observe that by (4.13),

$$G_N v = G_N \begin{pmatrix} v_1^{(E)} \\ v_2^{(E)} \end{pmatrix} = A_N \cdots A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_N(x_0) \\ a_{N-1} p_{N-1}(x_0) \end{pmatrix}. \quad (4.17)$$

Since G_N is invertible and that $a_n > 0$ for all n , $v = 0$ implies $p_N(x_0) = p_{N-1}(x_0) = 0$. This contradicts the fact that the zeros of $p_n(x)$ and $p_{n-1}(x)$ strictly interlace for all n and $x \in \mathbb{R}$. \square

In the following proposition, we are going to identify the trichotomy about the pair $(v_n^{(1)}, v_n^{(2)})$ in (4.16):

Proposition 4.2. *Let $v_1^{(E)}$ and $v_2^{(E)}$ be defined as in (4.13) and N be defined in (4.6). Then one of the following is true:*

- (1) *For some $E \geq N$, both $v_1^{(E)}$ and $v_2^{(E)}$ are non-zero.*
- (2) *For all $E \geq N$, $v_2^{(E)} \equiv 0$. In that case,*

$$p_j(x_0) = \left(\prod_{k=N}^{j-1} \lambda_k^+ \right) p_N(x_0). \quad (4.18)$$

- (3) *For all $E \geq N$, $v_1^{(E)} \equiv 0$. In that case,*

$$p_j(x_0) = \left(\prod_{k=N}^{j-1} \lambda_k^- \right) p_N(x_0). \quad (4.19)$$

Remark: Case (1) will be treated in Section 4.5 below.

Proof. Suppose for all $E \geq N$, $v_2^{(E)} \equiv 0$. By Proposition 4.1, $v_1^{(E)} \neq 0$ for all E . That is equivalent to

$$G_E \begin{pmatrix} v_1^{(E)} \\ 0 \end{pmatrix} = \begin{pmatrix} p_E(x_0) \\ a_{E-1} p_{E-1}(x_0) \end{pmatrix}, \quad \forall E \geq N. \quad (4.20)$$

Recall from (4.11) that $A_E = G_E D_E G_E^{-1}$. In other words, G_E is the change of basis matrix that maps the vector $(1, 0)$ to the eigenvector of A_E with eigenvalue λ_E^+ . Therefore,

$$\begin{pmatrix} p_{E+1}(x_0) \\ a_E p_E(x_0) \end{pmatrix} = A_E \begin{pmatrix} p_E(x_0) \\ a_{E-1} p_{E-1}(x_0) \end{pmatrix} = \lambda_E^+ \begin{pmatrix} p_E(x_0) \\ a_{E-1} p_{E-1}(x_0) \end{pmatrix}. \quad (4.21)$$

By an inductive argument, we obtain (4.18).

The proof for (3) is identical in nature except that we have

$$\begin{pmatrix} p_{E+1}(x_0) \\ a_E p_E(x_0) \end{pmatrix} = \lambda_E^- \begin{pmatrix} p_E(x_0) \\ a_{E-1} p_{E-1}(x_0) \end{pmatrix}. \quad (4.22)$$

instead of (4.21) because G_E maps $(0, 1)$ to the eigenvector of A_E with eigenvalue λ_E^- . This proves (4.19).

□

Now that we have asymptotic formulae for $p_n(x_0)$ for Cases (2) and (3), we are going to consider Case (1). In fact, if there exists E such that $v_1^{(E)}$ and $v_2^{(E)}$ are non-zero, the asymptotic formula for $p_n(x_0)$ will be quite similar to (4.18) or (4.19) except for the presence of error terms, which will be analyzed in the next section.

4.5. Several estimates. By Proposition 4.2, there are three cases. In this section we focus on Case (3): i.e., there exists an $E \geq N$ such that $v_1^{(E)}$ and $v_2^{(E)}$ are both non-zero.

Consider a fixed E . For convenience, we shall write

$$v_1 = v_1^{(E)}, v_2 = v_2^{(E)} \text{ and } v = v_E. \quad (4.23)$$

Proposition 4.3. *There is a constant C such that*

$$\|v(n+1) - D_{n+1}v(n)\| \leq C\|A_{n+1} - A_n\|(|u_n| + |w_n|). \quad (4.24)$$

Proof. Note that

$$v(n+1) - D_{n+1}v(n) = D_{n+1}(G_{n+1}^{-1}G_n - 1)v(n). \quad (4.25)$$

The goal is to give bounds for each of the components on the right hand side of (4.25). Since U is analytic on S_ϵ , on some compact subset of S_ϵ there exist constants $\eta_1, \eta_2 > 0$ such that

$$\|G_{n+1} - G_n\| \leq \|G\|\|U_{A_{n+1}} - U_{A_n}\| \leq \eta_1\|A_{n+1} - A_n\| \quad (4.26)$$

and

$$\|G_{n+1}^{-1}\| \leq \|G^{-1}\|\|U_{A_{n+1}}^{-1}\| \leq \eta_2. \quad (4.27)$$

Thus, for $\eta = \eta_1\eta_2$,

$$\|G_{n+1}^{-1}G_n - 1\| = \|G_{n+1}^{-1}(G_n - G_{n+1})\| \leq \eta\|A_{n+1} - A_n\|. \quad (4.28)$$

Furthermore,

$$\sup_{n \geq N} \|D_n\| = \sup_{n \geq N} |\lambda_n^+| < 2|\lambda^+| \quad (4.29)$$

and

$$\|v(n)\| = \left\| \begin{pmatrix} u_n L_n v_1 \\ w_n L_n v_2 \end{pmatrix} \right\| < C_1 |L_n| (|u_n| + |w_n|), \quad (4.30)$$

where $C_1 = \max\{|v_1|, |v_2|\} > 0$. By applying (4.28), (4.29) and (4.30) to (4.25), we finish the proof of Proposition 4.3. □

Proposition 4.4. *Let u_n and w_n be defined as in (4.14) and $v_1, v_2 \neq 0$. Then the following inequalities hold:*

(1) *There is a constant C_3 such that*

$$|u_{n+1} - u_n| \leq C_3\|A_{n+1} - A_n\|(|u_n| + |w_n|). \quad (4.31)$$

(2) *There is a constant C_4 such that*

$$\left| w_{n+1} - \frac{\lambda_{n+1}^-}{\lambda_{n+1}^+} w_n \right| \leq C_4\|A_{n+1} - A_n\|(|u_n| + |w_n|). \quad (4.32)$$

Proof. Recall that $L_{n+1} = \lambda_{n+1}^+ L_n$ and $v_n^{(1)} = L_n u_n v_1$. Since $v_1 \neq 0$,

$$|u_{n+1} - u_n| = \left| \frac{v_{n+1}^1 - \lambda_{n+1}^+ v_n^{(1)}}{v_1 L_{n+1}} \right| \leq \frac{\|w(n+1) - D_{n+1} w(n)\|}{|v_1 L_{n+1}|}. \quad (4.33)$$

Similarly, since $v_n^{(2)} = L_n w_n v_2$ and $v_2 \neq 0$,

$$\left| w_{n+1} - \frac{\lambda_{n+1}^-}{\lambda_{n+1}^+} w_n \right| = \left| \frac{v_{n+1}^{(2)} - \lambda_{n+1}^- v_n^{(2)}}{v_2 L_{n+1}} \right| \leq \frac{\|w(n+1) - D_{n+1} w(n)\|}{|v_2 L_{n+1}|}. \quad (4.34)$$

Apply Proposition 4.3 to the equations above to obtain (4.31) and (4.32). \square

The following lemma concerning u_n and w_n is central for this paper. As we shall see in the proof of Theorem 2.3, it implies the dichotomy between exponential decay and exponential growth of $p_n(x_0)$.

Lemma 4.1. *Let u_n and w_n be defined as in (4.14) and $v_1, v_2 \neq 0$. Then one of the following is true:*

- (1) *There exists a constant C such that $|u_n| \leq C|w_n|$. Moreover, given any $\epsilon > 0$, there exist an integer N_ϵ and a constant C_ϵ such that*

$$|w_n| \leq C_\epsilon \left(\left| \frac{\lambda^-}{\lambda^+} \right| + \epsilon \right)^n, \quad \forall n \geq N_\epsilon. \quad (4.35)$$

- (2) *$|w_n/u_n| \rightarrow 0$. Furthermore, $u_\infty = \lim_{n \rightarrow \infty} u_n$ exists and it is non-zero.*

Proof. There are two possible situations concerning u_n and w_n :

- (1) There exist a fixed integer K_0 and a constant C such that $|u_n| \leq C|w_n|$ for all $n \geq K_0$.
- (2) For any integer K and any constant H , there exists an integer $n_{K,H} \geq K$ such that $|u_{n_{K,H}}| > H|w_{n_{K,H}}|$.

Case (1): By (4.32), for $n \geq \max\{N, K_0\}$, there is a positive constant C_7 such that

$$|w_{n+1}| \leq \left(\left| \frac{\lambda_n^-}{\lambda_n^+} \right| + C_7 \|A_{n+1} - A_n\| \right) |w_n|. \quad (4.36)$$

Recall that $\|A_{n+1} - A_n\| \rightarrow 0$ and $\lambda_n^\pm \rightarrow \lambda^\pm$. Thus, given any $\epsilon > 0$, there exist an integer N_ϵ and a constant C_ϵ such that

$$|w_n| \leq C_\epsilon \left(\left| \frac{\lambda^-}{\lambda^+} \right| + \epsilon \right)^n \quad \forall n \geq N_\epsilon. \quad (4.37)$$

In other words, w_n decays exponentially fast. Hence, u_n also decays exponentially fast to zero given the inequality $|u_n| \leq C|w_n|$. This corresponds to (2a) of Lemma 4.1.

Case (2): Let $r_n = w_n/u_n$. The statement $r_n \rightarrow 0$ is by definition equivalent to proving that given any $\epsilon > 0$ there exists an integer J_ϵ such that $|r_j| < \epsilon$ for all $j \geq J_\epsilon$.

First, we show that both u_n and u_{n+1} are non-zero, as (4.39) below will involve u_n and u_{n+1} in the denominator.

By the assumption we can choose any H , we choose one such that $1/H < \epsilon$. Consider any fixed pair (K, H) (the choice of K will be made later in the proof). The existence of an integer

$n = n_{K,H} > K$ such that $|r_n| < 1/H = \epsilon$ is guaranteed, which implies $u_n \neq 0$. Furthermore, by the triangle inequality and (4.31),

$$\begin{aligned} \left| \frac{u_{n+1}}{u_n} \right| &\geq 1 - \left| \frac{u_{n+1} - u_n}{u_n} \right| \\ &\geq 1 - C_3 \|A_{n+1} - A_n\| (1 + |r_n|) > 0 \end{aligned} \quad (4.38)$$

which implies that u_{n+1} is also non-zero.

Next, observe that

$$\begin{aligned} &\left| r_{n+1} - \frac{\lambda_n^-}{\lambda_n^+} r_n \right| \\ &\leq \left| \frac{w_{n+1}}{u_{n+1}} - \frac{\lambda_n^-}{\lambda_n^+} \frac{w_n}{u_{n+1}} \right| + \left| \frac{\lambda_n^-}{\lambda_n^+} \right| \left| \frac{w_n}{u_{n+1}} - \frac{w_n}{u_n} \right| \\ &= \left| \frac{w_{n+1} - (\lambda_n^-/\lambda_n^+) w_n}{u_{n+1}} \right| + \left| \frac{\lambda_n^-}{\lambda_n^+} r_n \right| \left| \frac{u_n - u_{n+1}}{u_{n+1}} \right|. \end{aligned} \quad (4.39)$$

By (4.31) and (4.32), there is a positive constant C_8 such that

$$\begin{aligned} &\left| r_{n+1} - \frac{\lambda_n^-}{\lambda_n^+} r_n \right| \\ &\leq \frac{1 + |r_n| |\lambda_n^-/\lambda_n^+|}{|u_{n+1}|} C_8 \|A_{n+1} - A_n\| (|u_n| + |w_n|) \\ &= C_8 (1 + |r_n| |\lambda_n^-/\lambda_n^+|) \|A_{n+1} - A_n\| \frac{|u_n|}{|u_{n+1}|} (1 + |r_n|). \end{aligned} \quad (4.40)$$

By inverting (4.38),

$$\left| \frac{u_n}{u_{n+1}} \right| \leq \frac{1}{1 - C_3 \|A_{n+1} - A_n\| (1 + |r_n|)}. \quad (4.41)$$

Then we plug it into (4.40) to get

$$|r_{n+1}| \leq \left| \frac{\lambda_n^-}{\lambda_n^+} r_n \right| + \frac{C_8 (1 + |r_n| |\lambda_n^-/\lambda_n^+|) (1 + |r_n|)}{1 - C_3 \|A_{n+1} - A_n\| (1 + |r_n|)} \|A_{n+1} - A_n\|. \quad (4.42)$$

Since $\|A_{n+1} - A_n\| \rightarrow 0$, the second term on the right hand side of (4.42) can be arbitrarily small if n is sufficiently large. Hence, for any sufficiently large K , there exists $n > K$ such that $|r_{n+1}| < |r_n| < \epsilon$.

Inductively, we can apply the same argument to r_{n+1} to prove that $|r_{n+2}| < \epsilon$. As a result, $|r_j| < \epsilon$ for all large j . This proves $|w_n/u_n| \rightarrow 0$.

Next, we are going to show that $\lim_{n \rightarrow \infty} u_n$ exists. Divide (4.31) by $|u_n|$. Since $|r_n| \rightarrow 0$,

$$\left| \frac{u_{n+1}}{u_n} - 1 \right| \leq C \|A_{n+1} - A_n\| (1 + |r_n|) \rightarrow 0. \quad (4.43)$$

Moreover, since \log is analytic near 1, in an ϵ -neighborhood of 1 there is a constant E such that

$$|\log z| = |\log \zeta - \log 1| \leq E|z - 1|. \quad (4.44)$$

By (4.43),

$$\left| \log \left(\frac{u_{n+1}}{u_n} \right) \right| \leq C \|A_{n+1} - A_n\|. \quad (4.45)$$

Therefore, the series $\sum_{j=N}^{\infty} \log(u_{j+1}/u_j)$ is absolutely convergent. Furthermore, as we have seen in (4.38), $u_j \neq 0$ for all large j . Thus, $\log u_j$ is well-defined and the following limit

$$u_{\infty} := \lim_{n \rightarrow \infty} \log u_n = \lim_{n \rightarrow \infty} \sum_{j=p}^{n-1} (\log u_{j+1} - \log u_j) + \log u_p \quad (4.46)$$

exists and is finite.

This corresponds to the second part of (2b) and concludes the proof of Lemma 4.1. \square

4.6. Implications of Lemma 4.1. By Lemma 4.1, there are two possible situations:

Case (1). Suppose (2a) of Lemma 4.1 is true. We will prove that this corresponds to the case $\mu(x_0) > 0$.

First, observe that

$$\begin{pmatrix} p_n(x_0) \\ a_{n-1}p_{n-1}(x_0) \end{pmatrix} = T_n(x_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = G_n L_n \begin{pmatrix} u_n v_1 \\ w_n v_2 \end{pmatrix}. \quad (4.47)$$

Hence, given any $\epsilon > 0$, there exists a constant $K_{\epsilon} > 0$ such that

$$\begin{aligned} \left\| T_n(x_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| &\leq \|G_n\| \prod_{k=N+1}^n |\lambda_k^+| \left\| \begin{pmatrix} u_n v_1 \\ w_n v_2 \end{pmatrix} \right\| \\ &\leq K_{\epsilon} (|\lambda^+| + \epsilon)^n \left(\left| \frac{\lambda^-}{\lambda^+} \right| + \epsilon \right)^n. \end{aligned} \quad (4.48)$$

Since $|\lambda^-| < 1$ and $|\lambda^-/\lambda^+| < 1$, $p_n(x_0)$ goes to zero exponentially fast by (4.48).

It is well-known (see for example [5]) that

$$\mu(x_0) = \left(\lim_{N \rightarrow \infty} \sum_{n=0}^N p_n(x_0)^2 \right)^{-1}. \quad (4.49)$$

Hence, $\mu(x_0) > 0$ and we are just varying the weight of x_0 . It is easy to prove that by (2.8) and (2.9), both $a_n - \tilde{a}_n$ and $\tilde{b}_{n+1} - b_{n+1}$ go to 0 exponentially fast.

This is in agreement with Simon's result (Corollary 24.4 of [22]) that varying the weight of a pure point will result in exponentially small perturbation of the recurrence coefficients.

Case (2). Suppose (2b) of Lemma 4.1 is true. Let

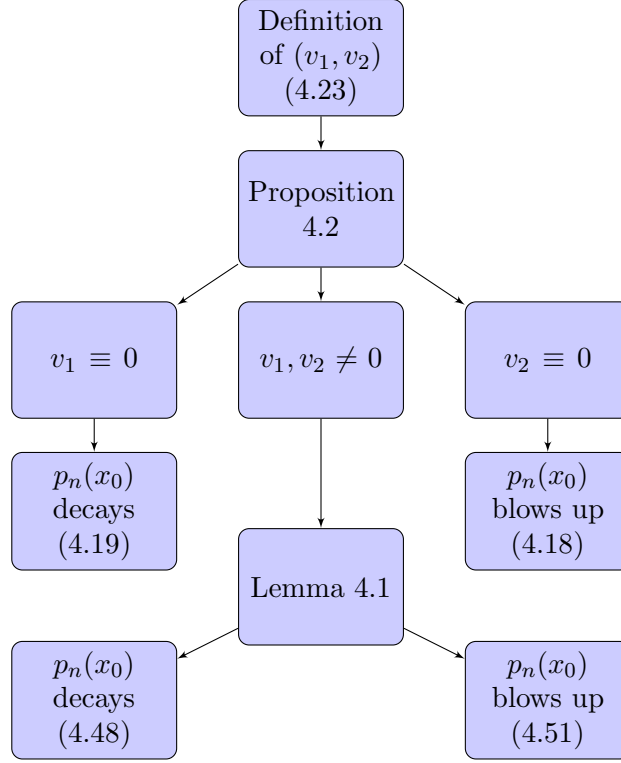
$$G_n = \begin{pmatrix} g_{1,n} & g'_{1,n} \\ g_{2,n} & g'_{2,n} \end{pmatrix} \rightarrow G = \begin{pmatrix} g_1 & g'_1 \\ g_2 & g'_2 \end{pmatrix}. \quad (4.50)$$

Since $p_n(x_0)$ is the first component of the vector $G_n L_n(u_n v_1, w_n v_2)^T$, we have

$$\begin{aligned} p_n(x_0) &= L_n(g_{1,n} u_n v_1 + g'_{1,n} w_n v_2) \\ &= L_n u_n (g_{1,n} + g'_{1,n} r_n v_2) \\ &= L_n(u_{\infty} g_1 v_1 + o(1)). \end{aligned} \quad (4.51)$$

The last equality holds because $r_n = u_n/w_n \rightarrow 0$ by (2b) of Lemma 4.1.

Here is a summary of the results in this section:



5. PROOF OF THEOREM 2.1

By the discussion in Section 4, $p_n(x_0)^2$ is either exponentially increasing or exponentially decaying towards zero. Moreover, by (4.49) above,

$$\mu(x_0) > 0 \iff \sum_{n=1}^{\infty} p_n(x_0)^2 < \infty. \quad (5.1)$$

Therefore, $\mu(x_0) > 0$ if and only if $p_n(x_0)^2$ is exponentially decaying towards zero, which corresponds to (4.19) and (4.48); $\mu(x_0) = 0$ if and only if $p_n(x_0)^2$ is exponentially increasing, which corresponds to (4.18) and (4.51).

6. PROOF OF THEOREM 2.2

Proof. Let $K_n(x, y)$ be the reproducing kernel of the measure μ , which is given by

$$K_n(x, y) = \sum_{j=0}^n p_j(x) p_j(y). \quad (6.1)$$

Since $\tilde{P}_n(x)$ is a polynomial of degree n ,

$$\begin{aligned} \tilde{P}_n(x) &= \int \tilde{P}_n(y) K_n(x, y) d\mu(y) \\ &= \int \tilde{P}_n(y) K_n(x, y) d\tilde{\mu}(y) - \gamma \tilde{P}_n(x_0) K_n(x, x_0). \end{aligned} \quad (6.2)$$

Moreover, $\tilde{P}_n(y)$ is orthogonal to all polynomials with degree $\leq n-1$ with respect to the inner product $\langle \cdot, \cdot \rangle_{d\tilde{\mu}}$. Therefore,

$$\begin{aligned} \int \tilde{P}_n(y) K_n(x, y) d\tilde{\mu}(y) &= \int \tilde{P}_n(y) p_n(y) p_n(x) d\tilde{\mu}(y) \\ &= p_n(x) \int \tilde{P}_n(y) p_n(y) d\tilde{\mu}(y) \end{aligned} \quad (6.3)$$

and that

$$\left\langle \tilde{P}_n(y), p_n(y) \right\rangle_{d\tilde{\mu}} = \left\langle \tilde{P}_n(y), \kappa_n y^n \right\rangle_{d\tilde{\mu}} = \frac{\kappa_n}{(\tilde{\kappa}_n)^2}. \quad (6.4)$$

Therefore, by (6.2) and (6.3),

$$\tilde{P}_n(x) = \left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 P_n(x) - \gamma \tilde{P}_n(x_0) K_n(x, x_0). \quad (6.5)$$

Now plug in $x = x_0$ into (6.5). Upon rearranging, we get

$$\tilde{P}_n(x_0) = \left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 \frac{P_n(x_0)}{1 + \gamma K_n(x_0, x_0)}. \quad (6.6)$$

Putting (6.6) into (6.5), we arrive at (2.4). In particular, note that both $P_n(x)$ and $\tilde{P}_n(x)$ are monic polynomials. Hence, by comparing the coefficients of x^n on each side of (2.4), we get

$$1 = \left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 \left(1 - \frac{\gamma p_n(x_0)^2}{1 + \gamma K_n(x_0, x_0)} \right) \quad (6.7)$$

$$= \left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 \frac{1 + \gamma K_{n-1}(x_0, x_0)}{1 + \gamma K_n(x_0, x_0)}. \quad (6.8)$$

Recall that $a_n = \kappa_{n-1}/\kappa_n$. This proves (2.7).

Next, we prove the formula for \tilde{b}_{n+1} . Let m_n be the coefficient of x^{n-1} in $P_n(x)$, i.e.,

$$P_n(x) = x^n + m_n x^{n-1} + \text{lower order terms}. \quad (6.9)$$

By the recurrence relation (1.6), b_{n+1} is given by the coefficient of x^n in $xP_n(x) - P_{n+1}(x)$, which can also be expressed as

$$b_{n+1} = m_n - m_{n+1}. \quad (6.10)$$

To prove formula (2.9) for \tilde{b}_{n+1} , we will compute \tilde{m}_n . By (2.4),

$$\tilde{m}_n = \left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 \left(m_n - \frac{\gamma P_n(x_0)}{1 + \gamma K_n(x_0, x_0)} [p_n(x_0) \kappa_n m_n + p_{n-1}(x_0) \kappa_{n-1}] \right). \quad (6.11)$$

The coefficients of m_n in (6.11) are given by

$$\left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 \left(1 - \frac{\gamma p_n(x_0)^2}{1 + \gamma K_n(x_0, x_0)} \right), \quad (6.12)$$

which is equal to 1 by (6.7). Therefore,

$$\begin{aligned} \tilde{m}_n &= m_n - \left(\frac{\kappa_n}{\tilde{\kappa}_n} \right)^2 \frac{\gamma P_n(x_0) p_{n-1}(x_0) \kappa_{n-1}}{1 + \gamma K_n(x_0, x_0)} \\ &= m_n - \frac{\gamma P_n(x_0) p_{n-1}(x_0) \kappa_{n-1}}{1 + \gamma K_{n-1}(x_0, x_0)}. \end{aligned} \quad (6.13)$$

The last equality follows from the expression of $(\kappa_n/\tilde{\kappa}_n)^2$ in (6.8).

This concludes the proof of Theorem 2.2. \square

7. PROOF OF THEOREM 2.2

We are going to separate the proof into two different cases.

Case (1): x_0 is a pure point of μ . This is the easier case. Since

$$\lim_{n \rightarrow \infty} K_n(x_0, x_0) = \mu(x_0)^{-1}, \quad (7.1)$$

it is clear by (2.8) that $t_n \rightarrow 1$ and $\tilde{a}_n \rightarrow a$ as $n \rightarrow \infty$. Furthermore, recall that $P_{n+1}(x_0) = p_{n+1}(x_0)/\kappa_{n+1}$ and $a_{n+1} = \kappa_n/\kappa_{n+1}$. Hence,

$$h_n := \frac{\gamma P_{n+1}(x_0) p_n(x_0) \kappa_n}{1 + \gamma K_n(x_0, x_0)} = \frac{\gamma a_{n+1} p_{n+1}(x_0) p_n(x_0)}{1 + \gamma K_n(x_0, x_0)} \rightarrow 0 \quad (7.2)$$

because $p_n(x_0) \rightarrow 0$ exponentially fast. Therefore,

$$\tilde{b}_{n+1} = b_{n+1} - h_{n-1} + h_n \rightarrow b \text{ as } n \rightarrow \infty. \quad (7.3)$$

It is trivial to prove (2.11) for this particular case so we shall omit the proof.

Case (2): x_0 is not a pure point of μ . To prove that $\tilde{a}_n \rightarrow a$, we are going to prove that $\lim_{n \rightarrow \infty} t_n$ exists by employing the following theorem:

Theorem 7.1 (Cesàro–Stolz [3]). *Let $(\Gamma_n)_{n \in \mathbb{N}}, (K_n)_{n \in \mathbb{N}}$ be two sequences of numbers such that K_n is strictly increasing and tends to infinity. If the following limit*

$$\lim_{n \rightarrow \infty} \frac{\Gamma_n - \Gamma_{n-1}}{K_n - K_{n-1}} \quad (7.4)$$

exists then it is equal to $\lim_{n \rightarrow \infty} \Gamma_n / K_n$.

Let $\Gamma_n = 1 + \gamma K_{n-1}(x_0, x_0)$ and $K_n = 1 + \gamma K_n(x_0, x_0)$. Since $\lambda_n^+ > 1$, $L_n \rightarrow \infty$. Hence, $p_n(x_0) \rightarrow \infty$, which also implies that K_n is an increasing sequence to infinity. This allows us to apply the Cesàro–Stolz Theorem to prove that

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} \frac{\Gamma_n - \Gamma_{n-1}}{K_n - K_{n-1}} = \lim_{n \rightarrow \infty} \frac{p_{n-1}(x_0)^2}{p_n(x_0)^2} = \frac{1}{\lambda^+}. \quad (7.5)$$

The last equality of (7.5) follows from the asymptotic formula (4.51) for $p_n(x_0)$. As a result, $\lim_{n \rightarrow \infty} \tilde{a}_n = \lim_{n \rightarrow \infty} a_n = a$.

Next, we are going to prove that $\lim_{n \rightarrow \infty} \tilde{b}_n = b$. We shall employ the Cesàro–Stolz Theorem again. Let

$$\Gamma_n = \gamma p_{n+1}(x_0) p_n(x_0) \text{ and } K_n = 1 + \gamma K_n(x_0, x_0). \quad (7.6)$$

By the asymptotic formula (4.51) for $p_n(x_0)$,

$$\frac{\Gamma_n - \Gamma_{n-1}}{K_n - K_{n-1}} = \frac{\gamma p_n(x_0) (p_{n+1}(x_0) - p_{n-1}(x_0))}{\gamma p_n(x_0)^2} \rightarrow \lambda^+ - \frac{1}{\lambda^+}. \quad (7.7)$$

Therefore,

$$\lim_{n \rightarrow \infty} h_n = \lim_{n \rightarrow \infty} \frac{\gamma a_{n+1} p_{n+1}(x_0) p_n(x_0)}{1 + \gamma K_n(x_0, x_0)} = a \left(\lambda^+ - \frac{1}{\lambda^+} \right) \quad (7.8)$$

and by (7.3), $\tilde{b}_n \rightarrow b$.

We are going to prove that the sequence $(\tilde{a}_n)_n$ is of bounded variation.

Since $\lim_{n \rightarrow \infty} \tilde{a}_n = a \neq 0$, for all large n there exists a constant C such that

$$|\tilde{a}_{n+1}^2 - \tilde{a}_n^2| = |\tilde{a}_{n+1} - \tilde{a}_n| |\tilde{a}_{n+1} + \tilde{a}_n| \geq C |\tilde{a}_{n+1} - \tilde{a}_n|. \quad (7.9)$$

Therefore, it is enough to prove that $(\tilde{a}_n^2)_n$ is of bounded variation. Furthermore, by the formula for \tilde{a}_n (2.7) and the fact that $\lim_{n \rightarrow \infty} t_n = 1/\lambda^{+2}$, it suffices to show that the sequence $(t_n)_n$ is of bounded variation.

To do that, we are going to show that the sequence $(\tau_n)_n$ is of bounded variation, where

$$\tau_n = \frac{p_n(x_0)^2}{\gamma^{-1} + K_n(x_0, x_0)}. \quad (7.10)$$

There are two cases to consider, $p_n(x_0)$ being in the form (4.18) or $p_n(x_0)$ being in the form (4.51).

If $p_n(x_0)$ is in the form (4.18),

$$\tau_n = \frac{L_n^2}{\gamma^{-1} + K_n(x_0, x_0)} \frac{p_N(x_0)^2}{\lambda_n^{+2}} \quad (7.11)$$

If $p_n(x_0)$ is in the form (4.51),

$$\tau_n = \underbrace{\frac{L_n^2}{\gamma^{-1} + K_n(x_0, x_0)}}_{(I)} \underbrace{u_n^2 (g_{1,n}v_1 + g'_{1,n}r_nv_2)^2}_{(II)}. \quad (7.12)$$

Observe that $p_N(x_0)^2/(\lambda_N^+)^2$ is of bounded variation, because $p_N(x_0)^2$ is a constant and $(1/(\lambda_n^+)^2)_n$ is of bounded variation. Moreover, $|g_{1,n+1} - g_{1,n}|$, $|g'_{1,n+1} - g'_{1,n}|$ and $|r_{n+1} - r_n|$ are of the order $O(\|G_{n+1} - G_n\|)$, which implies they are also of bounded variation. Therefore, given (7.11) and (7.12), it remains to show that $L_n^2/(\gamma^{-1} + K_n(x_0, x_0))$ is of bounded variation.

We will make use of the simple fact: given the equality

$$\frac{1}{y_n} - \frac{1}{y_{n+1}} = \frac{y_{n+1} - y_n}{y_{n+1}y_n}, \quad (7.13)$$

if $\lim_{n \rightarrow \infty} y_n = y \neq 0$, $(y_n)_n$ is of bounded variation if and only if $(1/y_n)_n$ is of bounded variation.

Hence, we will prove that $(\gamma^{-1} + K_n(x_0, x_0))/L_n^2$ is of bounded variation and its limit exist when n goes to infinity. To prove the latter, observe that by the Cesàro–Stolz Theorem,

$$\lim_{n \rightarrow \infty} \frac{L_{n+1}^2 - L_n^2}{K_{n+1}(x_0, x_0) - K_n(x_0, x_0)} = \frac{L_n^2 ((\lambda_{n+1}^+)^2 - 1)}{p_{n+1}(x_0)^2} = \ell ((\lambda^+)^2 - 1) \neq 0, \quad (7.14)$$

where ℓ is a non-zero constant whose value depends on whether $p_n(x_0)$ is in the form (4.18) or (4.51).

For the convenience of computation we will define a few more objects below. First, we let

$$\Lambda_n = \begin{cases} \lambda_n^+ & \text{if } n \geq N+1 \\ 1 & \text{if } 0 \leq n \leq N \end{cases}. \quad (7.15)$$

Then by (4.15), $L_n = \prod_{j=1}^n \Lambda_j$. Moreover, recall the definition of u_n in (4.14), which was only defined for $n \geq N$. For $0 \leq n \leq N$, let u_n w_n be defined implicitly by (4.14). The introduction of these objects will not affect the result of our computation.

Observe that

$$\frac{\gamma^{-1} + K_n(x_0, x_0)}{L_n^2} = \frac{1 + \gamma^{-1}}{L_n^2} + R_n, \quad (7.16)$$

where

$$R_n = \frac{1}{L_n^2} \sum_{j=1}^n p_j(x_0)^2 = \sum_{j=1}^n \frac{u_j^2 (g_{1,j}v_1 + g'_{1,j}r_jv_2)^2}{\lambda_{j+1}^+ \cdots \lambda_n^+}, \quad (7.17)$$

with the convention that $\lambda_{j+1}^+ \cdots \lambda_n^+ = 1$ when $j \geq n$.

Let

$$S_n = \frac{K_{n-1}(x_0, x_0)}{L_{n-1}^2} = \sum_{j=0}^{n-1} \frac{u_j^2 (g_{1,j} v_1 + g'_{1,j} r_j v_2)^2}{\lambda_{j+1}^+ \cdots \lambda_{n-1}^+}. \quad (7.18)$$

Then

$$\left| \frac{\gamma^{-1} + K_n(x_0, x_0)}{L_n^2} - \frac{\gamma^{-1} + K_{n-1}(x_0, x_0)}{L_{n-1}^2} \right| \leq \frac{2(1 + \gamma^{-1})}{(\lambda_{n-1}^+ \cdots \lambda_0^+)^2} + |R_n - S_n|. \quad (7.19)$$

Recall that $\lambda_n^+ \rightarrow \lambda^+ > 1$. Hence,

$$\sum_n \frac{1}{(\lambda_n^+ \cdots \lambda_0^+)^2} = O\left(\sum_n \frac{1}{(\lambda^+)^2}\right) < \infty. \quad (7.20)$$

Thus, the first sum on the right hand side of (7.19) is summable.

Next, observe that upon rearranging the indices of S_n in (7.18), we have

$$\begin{aligned} & |R_n - S_n| \\ &= \left| \sum_{j=1}^n \frac{u_j^2 (g_{1,j} v_1 + g'_{1,j} r_j v_2)^2}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} - \frac{u_{j-1}^2 (g_{1,j-1} v_1 + g'_{1,j-1} r_{j-1} v_2)^2}{(\lambda_j^+ \cdots \lambda_{n-1}^+)^2} \right| \\ &\leq \sum_{j=1}^n \frac{u_j^2 |e_j - e_{j-1}|}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} + \sum_{j=1}^n u_j^2 e_{j-1} \left| \frac{1}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} - \frac{1}{(\lambda_j^+ \cdots \lambda_{n-1}^+)^2} \right| \\ &\quad + \sum_{j=1}^n \frac{|u_j^2 - u_{j-1}^2| e_{j-1}}{(\lambda_j^+ \cdots \lambda_{n-1}^+)^2}, \end{aligned} \quad (7.21)$$

where

$$e_j = (g_{2,j} v_1 + g'_{2,j} r_j v_2)^2. \quad (7.22)$$

Now we proceed to estimate each of the sums on the last line of (7.21). Recall that $|u_j - u_{j-1}| = O(\|G_j - G_{j-1}\|)$. Moreover, $u_j \rightarrow u_\infty$. Hence, $\sum_j |u_j^2 - u_{j-1}^2| < \infty$. Therefore, for some constant C ,

$$\sum_{n=1}^{\infty} \sum_{j=1}^n \frac{|u_j^2 - u_{j-1}^2| e_{j-1}}{(\lambda_j^+ \cdots \lambda_{n-1}^+)^2} \leq C \left(\sum_{j=1}^{\infty} |u_j^2 - u_{j-1}^2| \right) \left(\sum_{j=1}^{\infty} \frac{1}{(\lambda^+)^{2j}} \right) < \infty. \quad (7.23)$$

Similarly,

$$\sum_{n=1}^{\infty} \sum_{j=1}^n \frac{u_j^2 |e_j - e_{j-1}|}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} < \infty \quad (7.24)$$

because

$$\begin{aligned} |e_j - e_{j-1}| &= O(|r_j - r_{j-1}|) + O(|g_{1,j} - g_{1,j-1}|) + O(|g'_{1,j} - g'_{1,j-1}|) \\ &= O(\|G_j - G_{j-1}\|). \end{aligned} \quad (7.25)$$

Finally, we consider the second sum on the right hand side of (7.21). Note that

$$|\lambda_j^{+2} - \lambda_n^{+2}| \leq C |\lambda_j^+ - \lambda_n^+| \leq C \sum_{k=j}^{n-1} |\lambda_{k+1}^+ - \lambda_k^+|, \quad (7.26)$$

where C is a positive constant independent of j and n . As a result,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{j=1}^n \left| \frac{1}{(\lambda_j^+ \cdots \lambda_{n-1}^+)^2} - \frac{1}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} \right| \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{|\lambda_j^{+2} - \lambda_n^{+2}|}{(\lambda_j^+ \cdots \lambda_n^+)^2} \\
&\leq C \sum_{n=1}^{\infty} \sum_{j=1}^n \frac{1}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} \sum_{k=j}^{n-1} |\lambda_{k+1}^+ - \lambda_k^+|.
\end{aligned} \tag{7.27}$$

Consider a fixed $k \in \mathbb{N}$. We count the coefficients of $|\lambda_{k+1}^+ - \lambda_k^+|$ in the last line of (7.27). From the equation, we know that $j \leq k < n$. Thus, the coefficient is

$$\begin{aligned}
& \sum_{n=k+1}^{\infty} \sum_{j=1}^k \frac{1}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} \\
&= \sum_{j=1}^k \left(\sum_{n=k+1}^{\infty} \frac{1}{(\lambda_{j+1}^+ \cdots \lambda_n^+)^2} \right) \\
&= \left(\sum_{j=1}^k \frac{1}{(\lambda_2^+ \cdots \lambda_j^+)^2} \right) \left(\sum_{n=k+1}^{\infty} \frac{1}{(\lambda_{k+1}^+ \cdots \lambda_n^+)^2} \right),
\end{aligned} \tag{7.28}$$

which is bounded above by a constant B independent of k . Therefore,

$$(7.27) \leq CB \sum_{k=1}^{\infty} |\lambda_{k+1}^+ - \lambda_k^+| < \infty. \tag{7.29}$$

Going back to (7.21), we conclude that $\sum_n |R_n - S_n| < \infty$. Hence, \tilde{a}_n is of bounded variation.

Finally, we are going to show that \tilde{b}_n is of bounded variation. Since a_n is of bounded variation, by (2.9), it suffices to show that $(h_n)_n$ is of bounded variation, h_n as defined in (7.2).

By (4.51),

$$h_n = \frac{a_{n+1} \lambda_{n+1}^+ L_n^2 u_{n+1} u_n}{\gamma^{-1} + K_n(x_0, x_0)} (g_{1,n+1} v_1 + g'_{1,n+1} r_{n+1} v_2) (g_{1,n} v_1 + g'_{1,n} r_n v_2). \tag{7.30}$$

Thus, it boils down to proving that $L_n^2/(\gamma^{-1} + K_n(x_0, x_0))$ is of bounded variation. To that end, we use the same argument following (7.10).

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